FALL 2022: MATH 790 HOMEWORK

The page numbers in each assignment below refer to those in the course textbooks. LADW refers to the text *Linear Algebra Done Wrong*

HW 1. Read Chapter 1 of LADW and work the following problems.

Let V be a vector space over the field F. Note that we are not assuming that V is finite dimensional.

(i) Let $\{u_1, \dots, u_n\} \subseteq V$ and set $U := \langle u_1, \dots, u_n \rangle$. Suppose $v_1, \dots, v_m \in U$ are linearly independent. Prove that $m \leq n$.

(ii) Use (i) to show that if V has a finite basis, then every basis of V has the same number of elements.

(iii) Assume that $F = \mathbb{C}$. Prove that V is also a vector space over $\mathbb{R}$, and assuming V is finite dimensional over $\mathbb{C}$, find the dimension of V as a vector space over $\mathbb{R}$ in terms of the dimension of V over $\mathbb{C}$.

(iv) Let $W \subseteq V$ be a subspace. Use Zorn's lemma to prove there exists a subspace $U \subseteq V$ maximal with respect to the property that $W \cap U = 0$.

HW 2. Let V be a vector space over the field F.

(i) Prove that if the dimension of V equals n, with $n > 0$, then there cannot exist a chain of subspaces $(0) \subsetneq W_1 \cdots \subsetneq W_n \subsetneq V$. Conclude that if $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ is an ascending chain of subspaces of V, then there exists $n_0 \geq 1$ such that $U_s = U_{n_0}$, for all $s \geq n_0$.

(ii) Suppose F is infinite. Prove that V is not the union of finitely many proper subspaces of V.

HW 3. This homework uses the notation from the second day of class, as it appears in the Daily Update from August 24. Let A be an $n \times n$ matrix with coefficients in the field F.

(i) Let $T : V \to W$ be a linear transformation and set $A = [T]_{B_V}^{B_W}$. For $v \in V$ let $[v]_{B_V}$ denote the $n \times 1$ column vector in F^n obtained as follows: If $v = a_1v_1 + \cdots + a_nv_n$, then $[v]_{B_V} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. The vector $[T(v)]_{B_W}$ in W is defined similarly. Prove that $[T(v)]_{B_W} = A \cdot [v]_{B_V}$.

(ii) Let E be an elementary matrix, i.e., an $n \times n$ matrix obtained from I_n by applying an elementary row operation. Prove that EA is obtained from A by apply the same elementary row operation to A.

(iii) Let E be an elementary matrix. Show that: (a) If E is type 1 with $\lambda \in F$, then E^{-1} is type 1 using λ^{-1}; (b) If E is type 2, then $E^{-1} = E$; (c) If E is type 3, obtained by $R_i + \lambda R_j$ applied to I_n, then E^{-1} is $R_i - \lambda R_j$ applied to I_n.

(iii) Prove that if A is any matrix, then there is a sequence of elementary row operations that put A into *reduced row echelon form.*

(iv) Show that if E is an elementary matrix corresponding to an elementary row operation of a given type, then E^t is an elementary matrix corresponding to a row operation of the same type.

HW 4. For these problems, you may use any of the properties of the determinant derived in class.

(i) Let A be an $n \times n$ matrix over $\mathbb{Q}$ such that every entry is ± 1. Prove that $|A|$ is divisible by 2^{n-1}.

(ii) Suppose that A and B are $(2k+1) \times (2k+1)$ matrices over $\mathbb{R}$ such that $AB = -BA$. Prove that A and B cannot both be invertible.

HW 5. 1. LADW, Chapter 3: 3.11, 3.12, 5.3, 5.5, 5.6.

2. Suppose $A \in \mathrm{M}_n(F)$, $T \in \mathcal{L}(V, V)$ and $p(x) \in F[x]$, the ring of polynomials with coefficients in F.

(i) Prove that $p(A) \in \mathrm{M}_n(F)$ and $p(T) \in \mathcal{L}(V, V)$.

(ii) Explain why a matrix in $\mathrm{M}_n(F[x])$ can be regarded as a polynomial with coefficients in $\mathrm{M}_n(F)$.

HW 6. Let V be a vector space of dimension n over the field F.

(i) Prove that the vector spaces $\mathcal{L}(V, V)$ and $\mathrm{M}_n(F)$ are isomorphic.
(ii) Using the Cayley-Hamilton theorem for matrices, prove that $\chi_T(T) = 0$, for all $T \in \mathcal{L}(V, V)$.
(iii) For $f(x) \in F[x]$, with s the degree of $f(x)$, prove that $|xI_s - C(f(x))| = f(x)$, where $C(f(x))$ is the companion matrix of $f(x)$.

HW 7. Prove the uniqueness statement in the division algorithm, i.e., prove that if $f(x), g(x), h(x), r(x), h_0(x), r_0(x)$ are in $F[x]$ and

$$g(x) = f(x)h(x) + r(x) = f(x)h_0(x) + r_0(x),$$

where $r(x)$, $r_0(x)$ are either zero or have degree less than the degree of $f(x)$, then $h(x) = h_0(x)$ and $r(x) = r_0(x)$.

HW 8. LADW, Chapter 5: 1.7, 1.8. And: Verify the inner product space axioms for Example 2 of the Daily Update for September 9.

HW 9. 1. Let $V = \mathrm{M}_2(\mathbb{R})$ with inner product $\langle A, B\rangle := \mathrm{tr}(A^tB)$. Find an orthonormal basis for the subspace $W := \langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$.

2. Let $P_2(\mathbb{R})$ denote the vector space of real polynomials having degree less than or equal to 2 with inner product $\langle p(x), q(x)\rangle := \int_0^1 p(x)q(x)\,dx$. Find an *orthogonal* basis for $P_2(\mathbb{R})$.

HW 10. Let V be a vector space over the field F.

1. Assume $v_1, \ldots, v_n \in V$ is a basis for V. For $1 \leq r \leq n$, set $W_1 := \langle v_1, \ldots, v_r\rangle$ and $W_2 := \langle v_{r+1}, \ldots, v_n\rangle$. Prove that $V = W_1 \oplus W_2$.

2. Suppose $V = W_1 \oplus \cdots \oplus W_t$ for subspaces $W_i \subseteq V$. Fix $1 \leq r \leq t$ and set $U_1 := W_1 + \cdots + W_r$ and $U_2 := W_{r+1} + \cdots + W_t$. Prove that $V = U_1 \oplus U_2$.

3. Assume that V is finite dimensional and $T \in \mathcal{L}(V, V)$. Consider the following scenario: $\mu_T(x) = p(x)q(x)$, where $p(x), q(x) \in F[x]$ have no common factor. Write W for the kernel of $p(T)$ and U for the kernel of $q(T)$. Prove that $V = W \oplus U$. For this you will need the following consequence of *Bezout's Principle*: There exist $a(x), b(x) \in F[x]$ such that $1 = a(x)p(x) + b(x)q(x)$.

HW 11. For the real symmetric matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix. Here, we mean that the columns of the diagonalizing matrix P should form an orthonormal basis for $\mathbb{R}^3$.

HW 12. 1. Let F be a field and $T_A : F^2 \to F^2$ be the linear transformation whose matrix with respect to the standard basis is $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Determine if T_A is diagonalizable over the fields: (a) $F = \mathbb{R}$, (b) $F = \mathbb{C}$, (c) $F = \mathbb{Z}_2$, and (d) $F = \mathbb{Z}_3$.

2. Let $T_B : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose matrix with respect to the standard basis is $B = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}$. Show that T_B is diagonalizable. Find an invertible 2×2 matrix P such that $P^{-1}BP$ has the eigenvalues of B down its diagonal.

3. Let $v_1', \ldots, v_n'$ be a basis for V and $P = (p_{ij})$ be an invertible $n \times n$ matrix. For each $1 \leq j \leq n$, write $v_j = p_{1j}v + 1 + \cdots + p_{nj}v_n'$. Prove that $v_1, \ldots, v_n$ is a basis for V.

HW 13. For A and B as in Homework 12, find an invertible matrices that diagonalize A and B.

HW 14. Let V denote the vector space of 2×2 real matrices with standard basis $E_{i,j}$ where $E_{i,j}$ is the 2×2 matrix with i, j entry equal to 1 and 0s elsewhere. Thus $\{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\}$ is a basis for V. Let tr denote the trace function as an element of V^*. Identifying V^* with the space of ordered 4-tuples with entries in F, write tr in terms of the dual basis $\{E_{1,1}^*, E_{1,2}^*, E_{2,1}^*, E_{2,2}^*\}$.

HW 15. 1. Let $T_C : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the standard basis is $C = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -7 & 4 \\ -2 & 4 & -7 \end{pmatrix}$. Find an orthonormal basis consisting of eigenvectors. Find an orthogonal matrix Q such that $Q^{-1}CQ$ has the eigenvalues of C down its diagonal.

2. Find a 2×2 matrix over $\mathbb{R}$ that is diagonalizable, but not orthogonally diagonalizable.

HW 16. 1. Let V be a finite dimensional inner product space over $\mathbf{C}$. Let $v, w \in V$ and suppose $\langle T(v), T(v)\rangle = \langle T^*(v), T^*(v)\rangle$ for all v in V. Prove that the imaginary parts of $\langle T(v), T(w)\rangle$ and $\langle T^*(v), T^*(w)\rangle$ are equal for all $v, w \in V$, by starting with the equation

$$\langle T(v - iw), T(v - iw)\rangle = \langle T^*(v - iw), T^*(v - iw)\rangle.$$

This completes the proof of the lemma from class stating that if $\langle T(v), T(v)\rangle = \langle T^*(v), T^*(v)\rangle$ for all v in V, then $\langle T(v), T(v)\rangle = \langle T^*(v), T^*(w)\rangle$, for all $v, w \in V$.

2. For V a finite dimensional inner product space over $\mathbf{C}$ and for $T \in \mathcal{L}(V, V)$, prove:

(i) $\text{kernel}(T^*) = (\text{range}(T))^\perp$.
(ii) $\text{kernel}(T)^\perp = (\text{range}(T^*)$.
(iii) $\text{kernel}(T) = (\text{range}(T^*))^\perp$
(iv) $\text{range}(T) = (\text{kernel}(T^*))^\perp$.

HW 17. Consider the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Show that A is a normal matrix and find: (a) An orthonormal basis $B \subseteq \mathbf{C}^3$ such that $[T]_B^B$ is diagonal and an orthonormal basis $D \subseteq \mathbb{R}^3$ such that $[T]_D^D = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix}$, for $\lambda, \alpha, \beta \in \mathbb{R}$ and $\beta > 0$.

HW 18. 1. Let A be an $m \times n$ matrix over $\mathbb{R}$ or $\mathbb{C}$. Prove that: (a) A^*A and AA^* have the same eigenvalues, counted with multiplicity and (b) A^*A and A have the same rank.

2. For each of the following matrices A, find the singular values of A and the unitary (or orthogonal) matrices Q and P (of the appropriate dimensions) such that $Q^*AP = \Sigma$, where Σ has the singular values of A down its diagonal: (a) $A = \begin{pmatrix} i & 2i \\ 3i & 6i \end{pmatrix}$; (b) $A = \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}$; (c) $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}$.

HW 19. Let $F[x]$ denote the ring of polynomials with coefficients in the field F.

(i) Let $p(x) \in F[x]$ be a non-constant irreducible polynomial. Prove that for any non-constant $f(x)$ in $F[x]$, the GCD of $p(x), f(x)$ is either $p(x)$ or 1.
(ii) Show that if $p(x)$ is irreducible over F and $p(x)$ divides $f(x) \cdot g(x)$, then $p(x)$ divides $f(x)$ or $p(x)$ divides $g(x)$. (Hint: Use (i) and Bezout's Principle.)
(iii) Prove that if $p_1(x) \cdots p_r(x) = q_1(x) \cdots q_s(x)$, and each $p_i(x), q_j(x)$ is irreducible over F, then $r = s$, and after re-indexing, $q_i(x) = \alpha_i \cdot p_i(x)$, for some $\alpha_i \in F$. In other words, the factorization property for polynomials in $F[x]$ is in fact a *unique factorization* property.

HW 20. 1. Consider $f(x) = x^4 + x^3 + x + 1$ and $x^4 + 2x$ in $\mathbb{Z}_2[x]$. Use the Euclidean algorithm to find the GCD of $f(x)$ and $g(x)$, then write this GCD as $a(x)f(x) + b(x)g(x)$, for some $a(x), b(x) \in \mathbb{Z}_2[x]$.

2. Give a detailed proof of the following: Assume V is a finite dimensional vector and $T : V \to V$ a linear operator on V. Suppose $V = W_1 \oplus \cdots \oplus W_r$, where each W_i is a T-invariant subspace of V. Suppose $B_i \subseteq W_i$ is basis of W_i and $A_i := [T|_{W_i}]_{B_i}^{B_i}$. Show that for $B = B_1 \cup \cdots \cup B_r$, $[T]_B^B$ is a block diagonal matrix with blocks $A_1, \ldots, A_r$.

HW 21. Consider the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ as an element of $M_2(\mathbb{R})$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(v) = Av$. Find a basis $B \subseteq \mathbb{R}^3$ such that the matrix of T with respect to B is block diagonal, with one block a 2×2 companion matrix and the other block a 1×1 matrix.

HW 22. 1. Find the invariant factor and elementary divisor rational canonical forms for the matrix $A = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

2. Suppose that A is a diagonalizable matrix, such that $\mu_A(x) = (x - \lambda_1) \cdots (x - \lambda_r)$. Prove that the elementary divisor rational canonical form of A is the diagonalization of A.

HW 23. 1. Let $A \in M_3(\mathbb{R})$. prove that $\mu_A(x)$ cannot be an irreducible polynomial of degree two.

2. For the matrix $B = \begin{pmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{pmatrix}$, find invertible 3×3 matrices P, Q such that $P^{-1}BP$ has the invariant factor RCF and $Q^{-1}BQ$ has the elementary divisor RCF. Note doing the econd part first may be easier.

3. For $A = \begin{pmatrix} c & 0 & -1 \\ 0 & c & 1 \\ -1 & 1 & c \end{pmatrix}$ find invertible matrices $P, Q \in M_3(F)$ such that $P^{-1}AP$ is the invariant factor RCF and $Q^{-1}AQ$ has the elementary divisor RCF.

HW 24. 1. For the matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, find an invertible matrix P so that $P^{-1}AP$ is a Jordan block.

2. Let $A, B \in M_n(F)$. Prove that the trace of AB equal the trace of BA. Conclude that if A and B are similar matrices, then A and B have the same trace.

HW 25. 1. Let $f(x) \in F[x]$ and C denote the companion matrix for $f(x)$. Prove that $\mu_C(x) = f(x)$.

2. Find the JCFs for the matrix A in problem 1 of Homework 23 in each of the following cases: $F = \mathbb{Q}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_{197}$. In each case find an invertible 3×3 matrix P over the appropriate field such that $P^{-1}AP$ is the relevant JCF.

HW 26. 1. Find all possible Jordan canonical forms for 9×9 matrices A over $\mathbb{C}$ whose minimal polynomial is $\mu_A(x) = (x-2)^2(x+i)^2(x-i)^2$.

2. Suppose $T \in V$ satisfies $\mu_T(x) = x^4$ and $V = \langle T, v_1 \rangle \oplus \langle T, v_2 \rangle$, where $\langle T, v_1 \rangle$ has basis $\{v_1, T(v_1), T^2(v_1), T^3(v_1)\}$ and $\langle T, v_2 \rangle$ has basis $\{v_2, T(v_2))\}$. Prove the following:

(i) $\{T^3(v_1), T(v_2)\}$ is a basis for the kernel of T

(ii) $\{T^2(v_1), T^3(v_1), v_2, T(v_2)\}$ is a basis for the kernel of T^2.

(iii) $\{T(v_1), T^2(v_1), T^3(v_1), v_2, T(v_2)\}$ is a basis for the kernel of T^3.

HW 27. Suppose A is a 14×14 JCF matrix with Jordan blocks $J(\lambda, 3), J(\lambda, 3), J(\lambda, 3), J(\lambda, 2), J(\lambda, 2), J(\lambda, 1)$. Verify the formulas for the number and sizes of the Jordan blocks by calculating the dimensions of null spaces of $(A - \lambda I), (A - \lambda I)^2, (A - \lambda I)^3$.

HW 28. Find at least four cube roots of the matrix $A = \begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}$. For this, you can use the fact, that if $\omega = e^{\frac{2\pi i}{3}}$, then $1, \omega, \omega^2$ are distinct cube roots of 1. Are there infinitely many cube roots of A? Find a formula for A^{2022}.

HW 29. 1. For $p = 3$, calculate $p_2(x), p_5(x)$, for $p_n(x)$ as in the proposition from the lecture of November 16. Then do the same for $p = 5$. Hint: Use the Taylor expansions of $(1+x)^3$ and $(1+x)^5$ about $x = 0$.

2. Find three cube roots of $A = \begin{pmatrix} 9 & -25 \\ 4 & -11 \end{pmatrix}$.

HW 30. For the matrix in HW 29, problem 2, find e^A and e^{At} and solve the system of differential equations:

$$x_1'(t) = 9x_1(t) - 25x_2(t)$$
$$x_2'(t) = 4x_1(t) - 11x_2(t).$$

HW 31. Find e^A and e^{At}, for the matrix $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$.

HW 32. 1. Let U, W be subspaces of the vector space V. Prove that $(U + W)/W$ is isomorphic to $U/(U \cap W)$. Hint: Find a well-defined surjective linear transformation from $U \to (U + W)/W$ and then apply the First Isomorphism Theorem.

2. Let V and U be vector spaces and $W \subseteq V$ a subspace. Set $K := \{f \in \mathcal{L}(V, U) \mid W \subseteq \text{kerne}(f)\}$. Show that K is a subspace of $\mathcal{L}(V, U)$ and $\mathcal{L}(V, U)/K \cong \mathcal{L}(V/W, U)$.

HW 33. 1. Give a detailed proof of the third isomorphism theorem stated in the lecture of November 30.

2. For vector spaces V and U over the field F, prove that $V \otimes_F U \cong U \otimes_F V$.

HW 34. Let V and U be vector spaces and suppose $\{v_\alpha\}_{\alpha \in A}$ is a basis for V and $\{u_\beta\}_{\beta \in B}$ is a basis for U. Fix a basis elements v_{α_1} and u_{β_1} from each basis. Now write a typical $v \in V$ as $v = c_1 v_{\alpha_1} + \Sigma_{\alpha \neq \alpha_1} c_\alpha v_\alpha$ and a typical element $u \in U$ as $u = d_1 u_{\beta_1} + \Sigma_{\beta \neq \beta_1} d_\alpha u_\beta$. Note c_1 and d_1 could be zero. Define $h : V \times U \to F$ by $h(v, u) := c_1 d_1$. Show that h is bilinear.